step size pyramids are finite. We built pyramids in decimal (base 10), why not try another base (e.g., binary)?

In all cases, we would be glad to hear of your results.

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PRIME TREES

RAY C. ELLIS, JR.
64 Randall Road
Berlin, MA 01503

FRED M. ELLIS
Department of Physics
Wesleyan University
Middletown, CT 06459

Problem

The generation of various types of "snowball primes" has been extended for certain starting numbers [1-5]. Because the numbers described here are extensive and more properly resemble a tree, they will be referred to as trees with roots, a trunk and branches. The starting number, or trunk, can be any number and branches and roots are formed by appending single digits to the left (branch) or right (root) of the number eliminating non-prime numbers. Obviously, the digits 0, 2, 4, 5, 6, and 8 will not propagate any roots and we also eliminate the digit 0 from the branches to avoid the trivial branches sequences such as 7, 07, 007, 0007, etc. An example of part of the tree for the trunk number 27 is given in Figure 1. All of the root structure and only the start of the branch structure is shown. Terminations are highlighted by dots.

Any trunk number thus has associated with it a complete tree structure. The roots will be significantly smaller than the branches due to the restriction of digit choices to 1, 3, 7, and 9, and are expected to terminate fairly rapidly. The propagation of branches, on the other hand, appears at first to expand without bound. In this article we report on our investigation into the size and structure of these trees for single digit trunks.

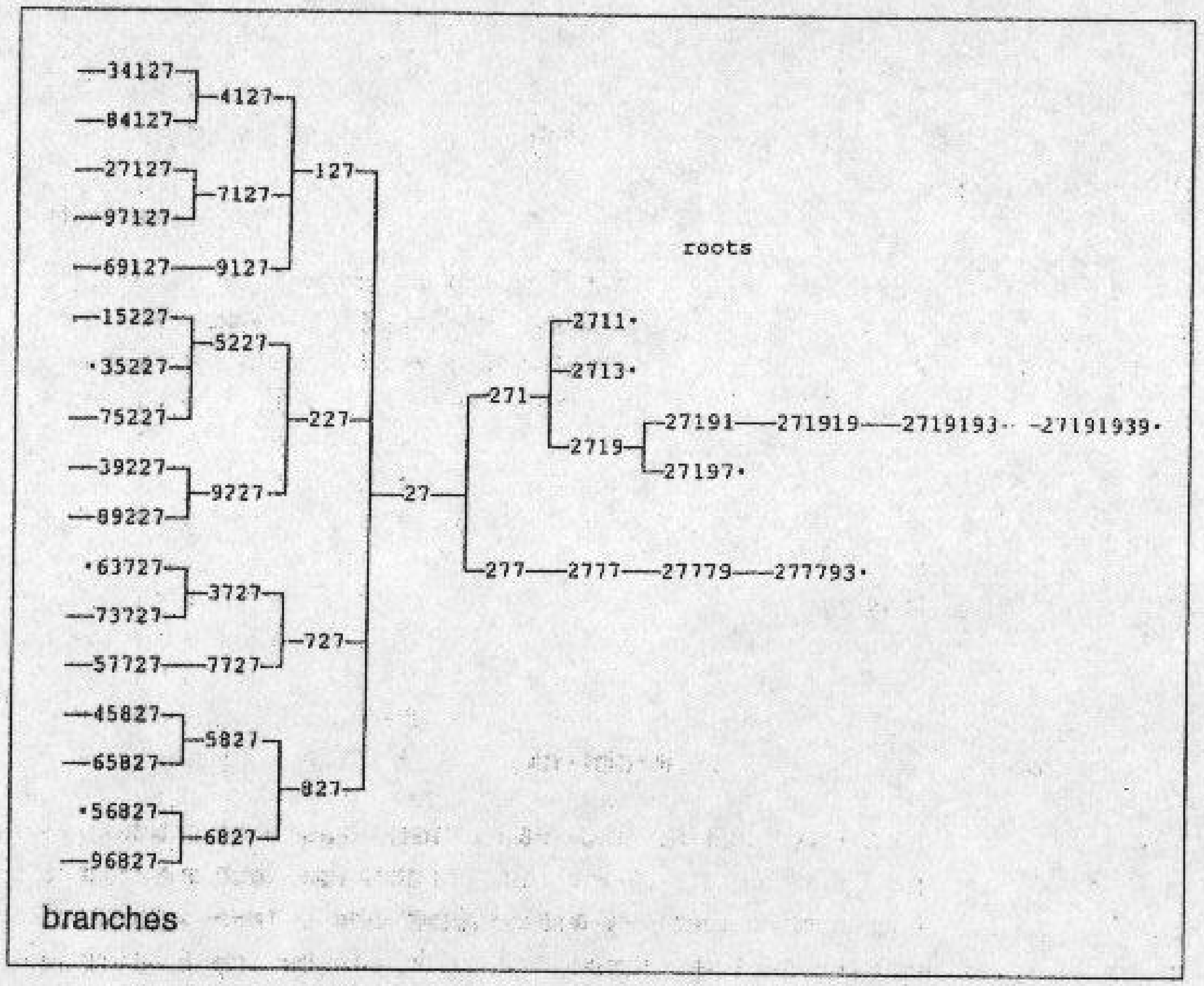

Figure 1. Part of a prime tree based on the number 27.

Theory

We first estimate the growth of the trees based on statistical arguments. Suppose at a level where there are d digits there are n_d branches or roots. The number of new appendages to test will be qn_d where $q = 9$ for branches (1–9) and $q = 4$ for roots (digits 1, 3, 7, and 9). The average number of these that will be prime will depend on the density of primes in the vicinity of numbers with d digits, and how many possible choices there are. The total number of primes with $d + 1$ digits is

$$\pi(10^{d+1}) - \pi(10^d) \qquad (1)$$

where $\pi(m)$ gives the number of primes $\leq m$. The pool of all allowed numbers with $d + 1$ digits is restricted to those ending in 1, 3, 7, and 9, so the total possible numbers are

$$\frac{4}{10}\left(10^{d+1} - 10^d\right) \qquad (2)$$

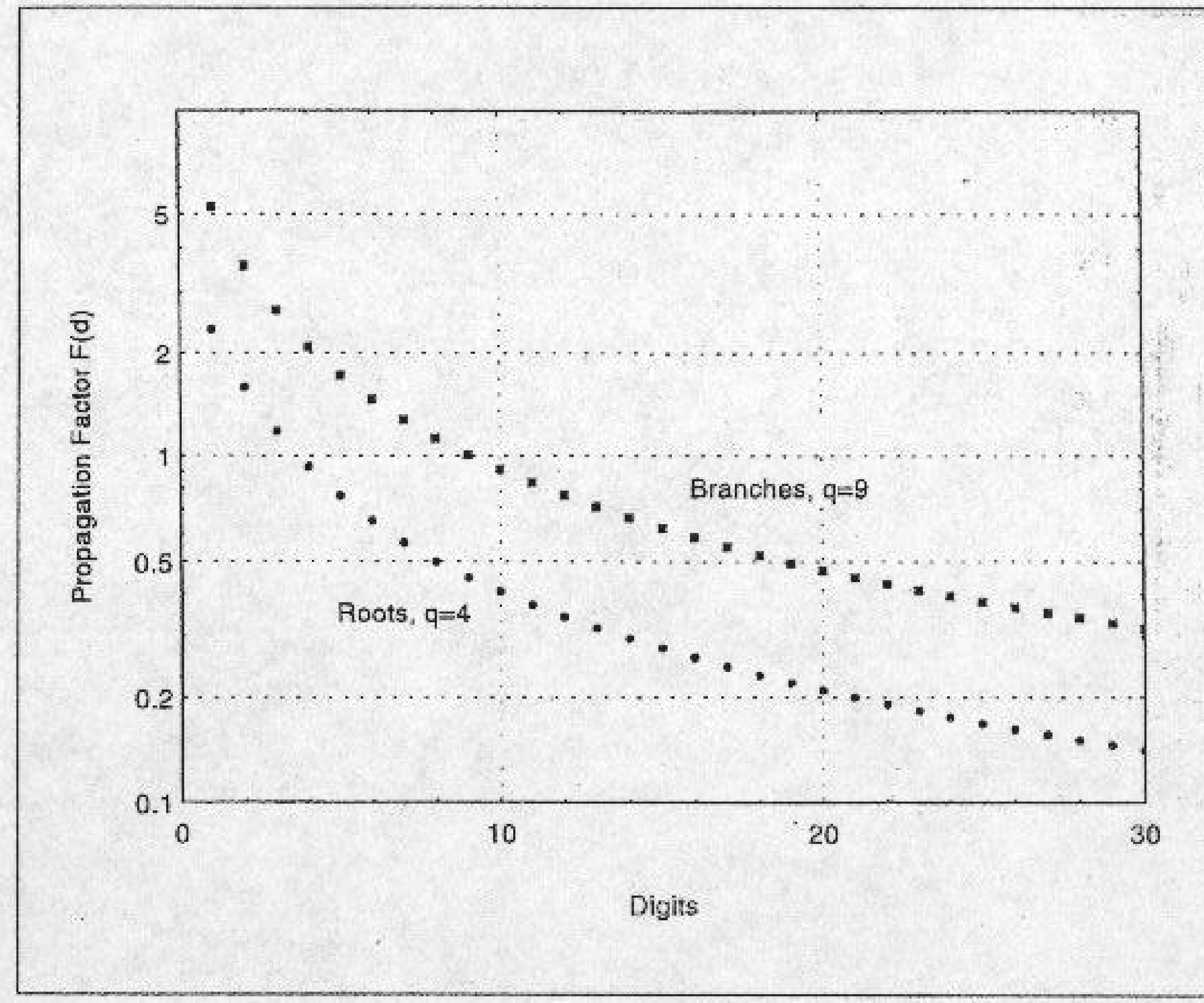

Figure 2. Appendage propagation factor.

and the ratio of Equation (1) to Equation (2) is the probability that an appendage will be prime. The surviving number of prime appendages with $d + 1$ digits is thus approximately qn_d times this ratio, or

$$n_{d+1} = \frac{q}{4} \cdot \left[\frac{\pi(10^{d+1}) - \pi(10^d)}{10^d - 10^{d-1}} \right] \cdot n_d \equiv F(d) \cdot n_d \,. \qquad (3)$$

Figure 2 shows this propagation factor $F(d)$ for both branches and roots calculated using tabulated values of $\pi(m)$ for $m < 10^8$ and the integral logarithm approximation

$$\pi(m) = \int_2^m \frac{dt}{ln(t)} \qquad (4)$$

for larger m. The structure of a one-digit-trunk tree can then be modeled by starting with $n_1 = 1$ and successively multiplying by the propagation factor as in Equation (3). It is clear from the behavior of $F(d)$ in Figure 2 that the structure, root, or branch should expand until the propagation factor becomes less than one,

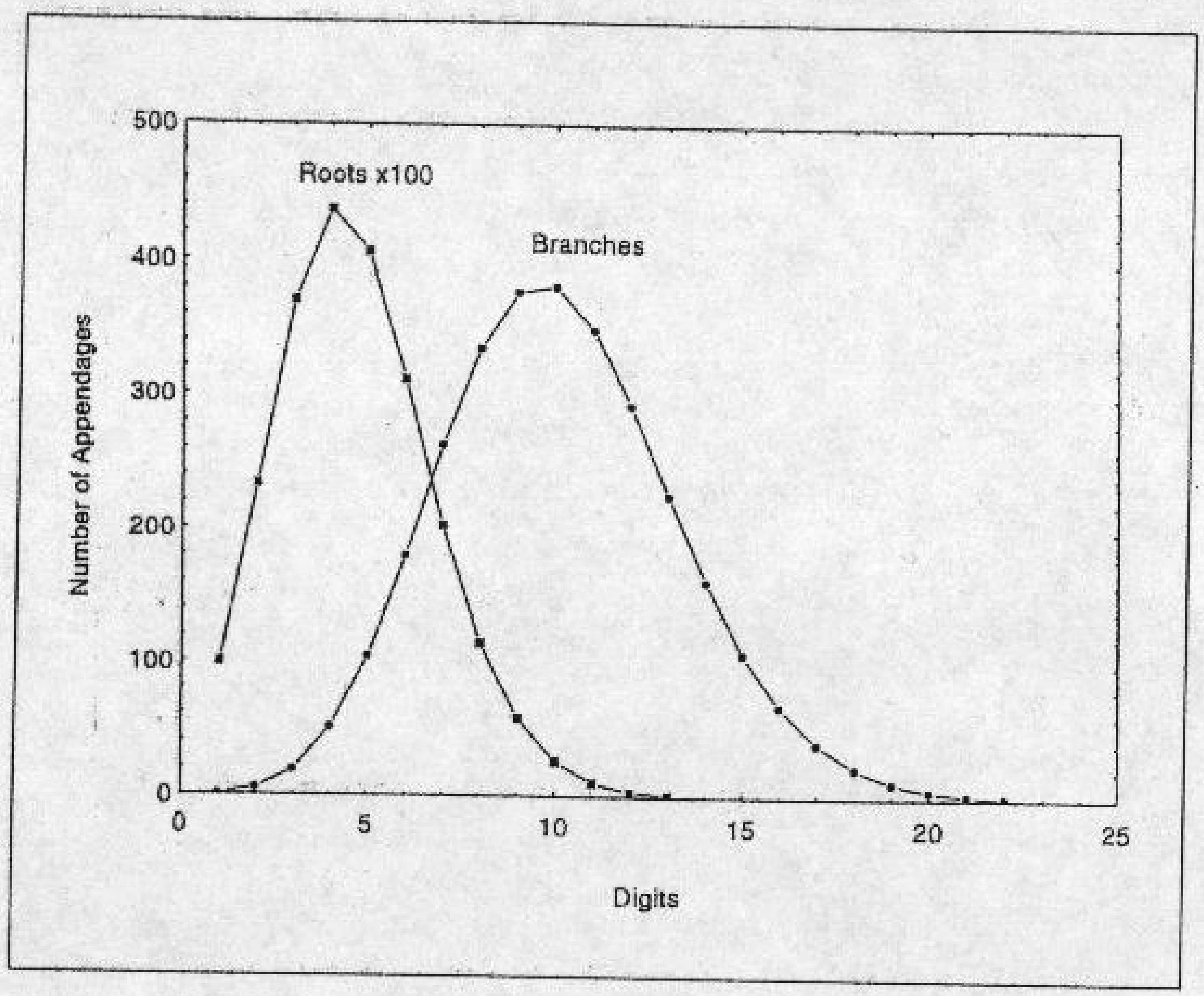

Figure 3. Expected root and branch population.

at which point it turns around and shrinks. This turn-around should occur at four digits for the roots and ten digits for the branches. Figure 3 shows the predicted number of branches and roots expected for the typical tree. Note that the number of roots is shown magnified by a factor of 100.

The compounding effect of the propagation factor generates a branch structure much larger than the root structure, as expected. The expected number of branches grows to around 380 at its peak and does not decay away until twenty-two digits, whereas the roots should grow to four or five digits and propagate to only eight digits. Notice that even though the branch propagation factor remains about one-third in this range, the branch does not survive long because of the compounding effect of the multiplications by $F(d)$.

Tree Generation

A *Mathematica* program was written to implement the branch or root growing procedure [6]. In the following discussion, reference is made only to branches—the root procedure differs only by appending digits to the right instead of the left.

Table 1. Branches

	Number of Branches (Ends)							
Digit	Trunk = 1		Trunk = 3		Trunk = 7		Trunk = 9	
1	1	(0)	1	(0)	1	(0)	1	(0)
2	5	(0)	6	(0)	5	(0)	5	(0)
3	18	(0)	22	(1)	17	(0)	14	(0)
4	45	(3)	52	(3)	47	(4)	42	(4)
5	81	(9)	103	(11)	89	(10)	86	(13)
6	141	(24)	176	(32)	150	(29)	135	(32)
7	206	(43)	230	(57)	199	(57)	177	(33)
8	264	(83)	283	(78)	238	(70)	229	(68)
9	270	(84)	281	(101)	264	(83)	256	(92)
10	266	(91)	260	(96)	257	(93)	239	(103)
11	245	(118)	218	(89)	230	(99)	188	(93)
12	188	(75)	173	(65)	181	(79)	134	(58)
13	154	(67)	137	(64)	139	(53)	111	(52)
14	123	(66)	95	(55)	117	(61)	74	(46)
15	77	(41)	47	(25)	70	(34)	34	(15)
16	46	(26)	26	(18)	46	(21)	22	(10)
17	25	(13)	9	(5)	33	(17)	15	(6)
18	13	(7)	5	(4)	19	(11)	12	(4)
19	8	(7)	2	(1)	11	(6)	9	(3)
20	1	(1)	1	(1)	5	(0)	8	(6)
21	0	(0)	0	(0)	5	(3)	2	(0)
22	0	(0)	0	(0)	4	(1)	3	(2)
23	0	(0)	0	(0)	3	(2)	1	(1)
24	0	(0)	0	(0)	1	(1)	0	(0)
25	0	(0)	0	(0)	0	(0)	0	(0)
Totals	2177	758	2127	706	2131	734	1797	641

The tree branch tips are maintained as a list of numbers. An operator is defined to pass through the list, operating on each number as follows: The number is checked for prime appendages. If it has none, it is left alone and the next number in the list is considered. If it has appendages, then the number is replaced by all of its surviving appendages and the operation continues at the first of the new appendage numbers. Thus, the list expands in front of the operator and movement past list elements occurs only at branch ends. When the operator finally passes through the list, the tree branch system is complete, being defined by the list of branch ends.

Table 2. Roots

Digit	Number of Roots (Root Ends)																	
	Trunk = 1		Trunk = 2		Trunk = 3		Trunk = 4		Trunk = 5		Trunk = 6		Trunk = 7		Trunk = 8		Trunk = 9	
1	1	(0)	1	(0)	1	(0)	1	(0)	1	(0)	1	(0)	1	(0)	1	(0)	1	(0)
2	4	(0)	2	(0)	2	(0)	3	(0)	2	(1)	2	(0)	3	(0)	2	(1)	1	(0)
3	10	(2)	3	(0)	5	(1)	5	(2)	2	(1)	5	(0)	4	(1)	1	(1)	2	(1)
4	12	(2)	5	(1)	6	(2)	6	(2)	1	(0)	8	(4)	4	(1)	0	(0)	1	(1)
5	14	(7)	5	(2)	5	(3)	7	(6)	2	(0)	6	(3)	3	(1)	0	(0)	0	(0)
6	10	(5)	3	(0)	2	(1)	1	(0)	2	(1)	5	(2)	5	(3)	0	(0)	0	(0)
7	5	(4)	3	(1)	1	(0)	1	(1)	1	(0)	3	(3)	3	(2)	0	(0)	0	(0)
8	1	(2)	2	(2)	1	(1)	0	(0)	1	(1)	0	(0)	1	(1)	0	(0)	0	(0)
9	1	(0)	0	(0)	0	(0)	0	(0)	0	(0)	0	(0)	0	(0)	0	(0)	0	(0)
10	2	(2)	0	(0)	0	(0)	0	(0)	0	(0)	0	(0)	0	(0)	0	(0)	0	(0)
11	0	(0)	0	(0)	0	(0)	0	(0)	0	(0)	0	(0)	0	(0)	0	(0)	0	(0)
Total	60	24	24	6	23	8	24	11	12	4	30	12	24	9	4	2	5	2

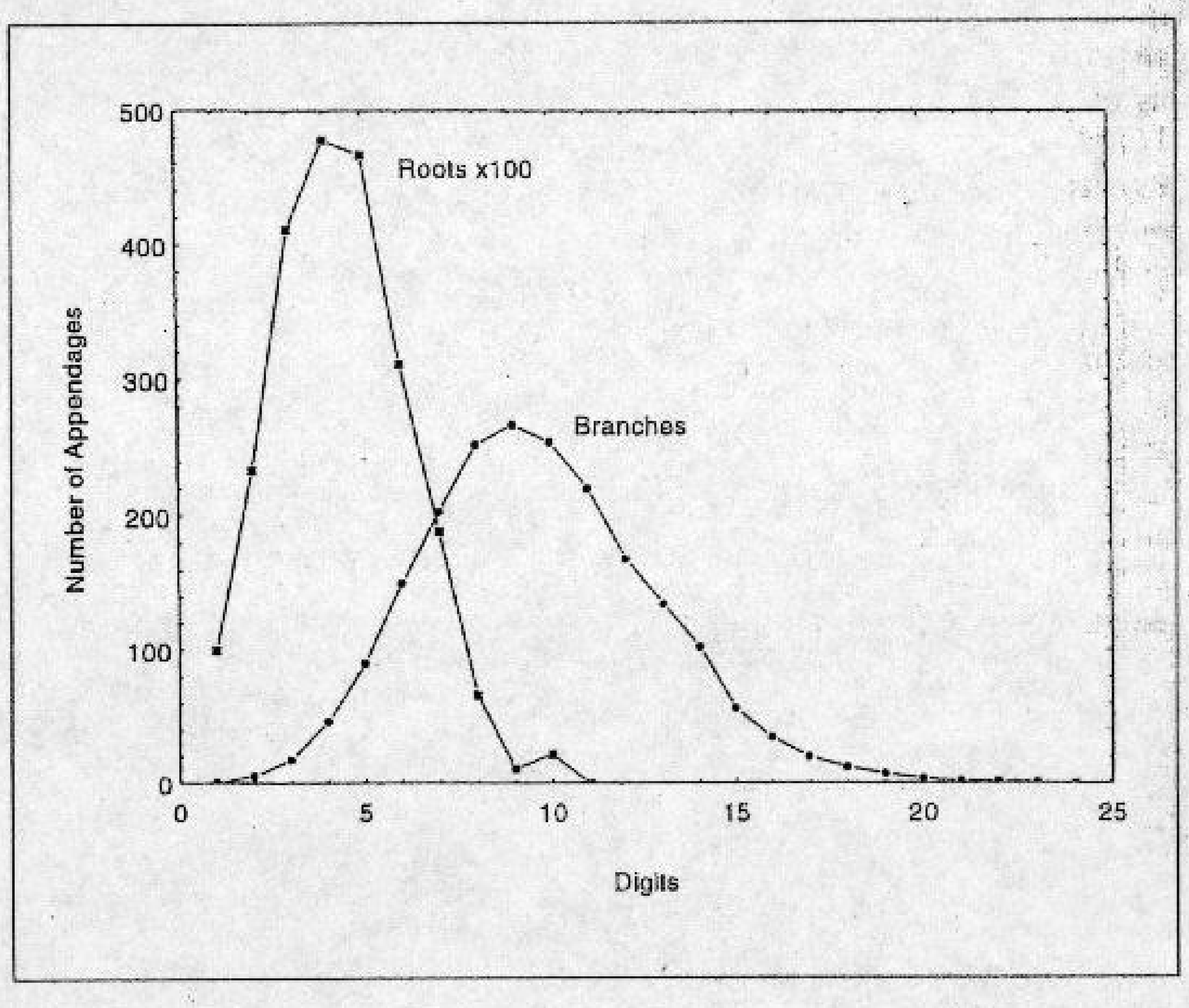

Figure 4. Actual branch and root populations.

Results

Branch structures were generated by the above prescription starting with the single digit trunks 1, 3, 7, and 9, and root structures for the single digit trunks 1 through 9. The results are presented in Tables 1 and 2 as the number of appendages propagated with each digit and the number of those that are terminal (branch/root ends). The averages of these results over the trunk digits are shown in Figure 4 (number) and Figure 5 (ends).

There are two deviations form the predicted behavior that are worth pointing out. First, the actual branch populations are about 30 percent smaller than expected in the vicinity of the peak (compare Figure 4 to Figure 3). This is not a consequence of "bad luck" from statistical fluctuations near the beginning. In fact, only the second digit would be capable of such statistics, and trunks 1, 3, 7, and 9 had remarkably good luck there. A careful comparison shows that this deviation results from a propagation factor that is about 95 percent of the theoretical $F(d)$ through the first fifteen digits. This deviation apparently applies only to the branches, but not the roots.

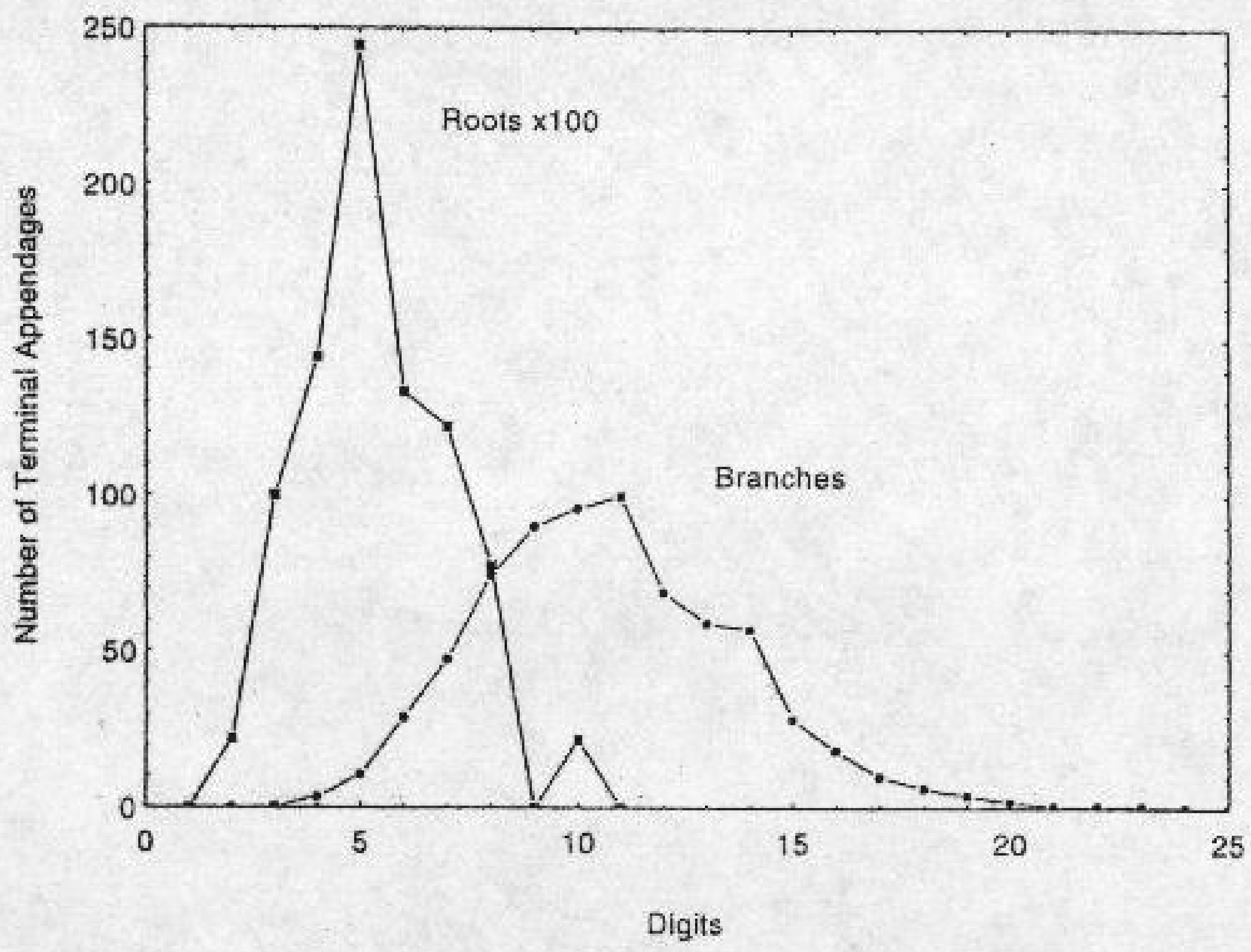

Figure 5. Root and branch ends.

The second anomaly is seen in Figure 5 as two prominent shoulders in the number of terminal branches at eleven and fourteen digits, representing respectively about 12 percent and 22 percent more terminations than expected by the distribution of primes. Again, this occurs at a digit level where chance is not a sufficient explanation. It is also a feature individually noticeable in all four trees.

We are unable to think of any reasons for either of these deviations. The first, involving an overall factor, is more likely to be the result of our omission of some obscure property of the prime distributions than the second. Having dealt with these deviations from our expectations, we now turn to some of the interesting numerical features of the single digit trunks investigated.

Of the 2839 branch terminations, there is only one of three digits, 773, and the fourteen four-digit terminations are:

3631	3373	3947	2389
4241	4643	6397	3919
6971	5113	6967	7229
		7937	9479

As expected, all branches finally die out in the vicinity of twenty-two digits. Those that extend to at least twenty digits are listed in Table 3, with the single largest branch represented by the 24-digit number 357686312646216567629137. It is fortunate that this number also has a single-digit prime trunk, 7, making it unique as the largest number that remains a prime if any number of its higher significant digits are broken off or truncated. This number was first found by Caldwell in his paper "Truncatable Primes" [7].

The shortest root structures are 53 and 89. The corresponding numbers for the longest roots are not blessed by a largest and further clouded by the interpretation of the trunk digit "1" as a prime: 1979339333 and 1979339339, at ten digits, remain prime if any number of their lower significant digits are broken off. If "1" is not included in the primes, there are five remaining 8-digit longest roots: 23399339, 29399999, 37337999, 59393339, and 73939133 with this property. Of course, if multiple digit trunks are included, longer sequences of either branches or roots undoubtedly exist.

Table 3. Notable Branches

(Digits)	Termination Prime
(20)	89726156799336363541
(20)	36484957213536676883
(21)	367986315421273233617
(23)	96686312646216567629137
(24)	357686312646216567629137
(23)	95918918997653319693967
(22)	9918918997653319693967
(21)	315396334245663786197
(21)	666276812967623946997
(22)	3863331219384687683719
(22)	9863331219384687683719
(20)	89127398757668667829
(20)	99127398757668667829
(20)	43989481279639897829
(20)	49989481279639897829
(20)	79989481279639897829
(23)	78738315342618762345659
(20)	75769956373951575479

Summary

We have investigated the idea of a prime tree in which any number, serving as a trunk, has an associated branch and root structure. Primes formed when single digits are appended to the left (branches) or right (roots) propagate the tree. The list of termination primes identifies either the branch or root system. The set of prime branches generated from single digit trunks is closed and consists of 8232 primes, of which 2839 are terminations. The set of prime roots of single digit trunks is less numerous than branches with 206 primes, of which 78 are terminations.

Both branches and roots grow out to a size consistent with the distribution of primes, although the total numbers of branches appears to deviate from what is statistically expected.

The number 357686312646216567629137 is the longest branch originating from a single digit trunk. It is the largest sequence of primes generated by successively appending single digits to the left.

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LETTER TO THE EDITOR:
Polystick Update

In [1], George Jelliss gave some historical information on polysticks and referred to unpublished work by H. D. Benjamin and T. R. Dawson which predates my paper on polysticks [2] by some forty years. I would like to add a little more historical data and present some more recent information including the enumeration of solutions to the 5 × 5 square problem.

I originally devised polysticks as an extension of polyominoes while I was a postgraduate student in the Department of Engineering Science at Oxford University in about 1970, and most of the material in [2] dates from about this period. The set of sixteen order-4 polysticks or "tetrasticks" (see Figure 1) seemed to have the greatest potential for challenging puzzles and was the main focus of attention. Although I considered publication at the time, the project was laid to one side as the task of writing a thesis took priority and was only taken up again some twenty years later.

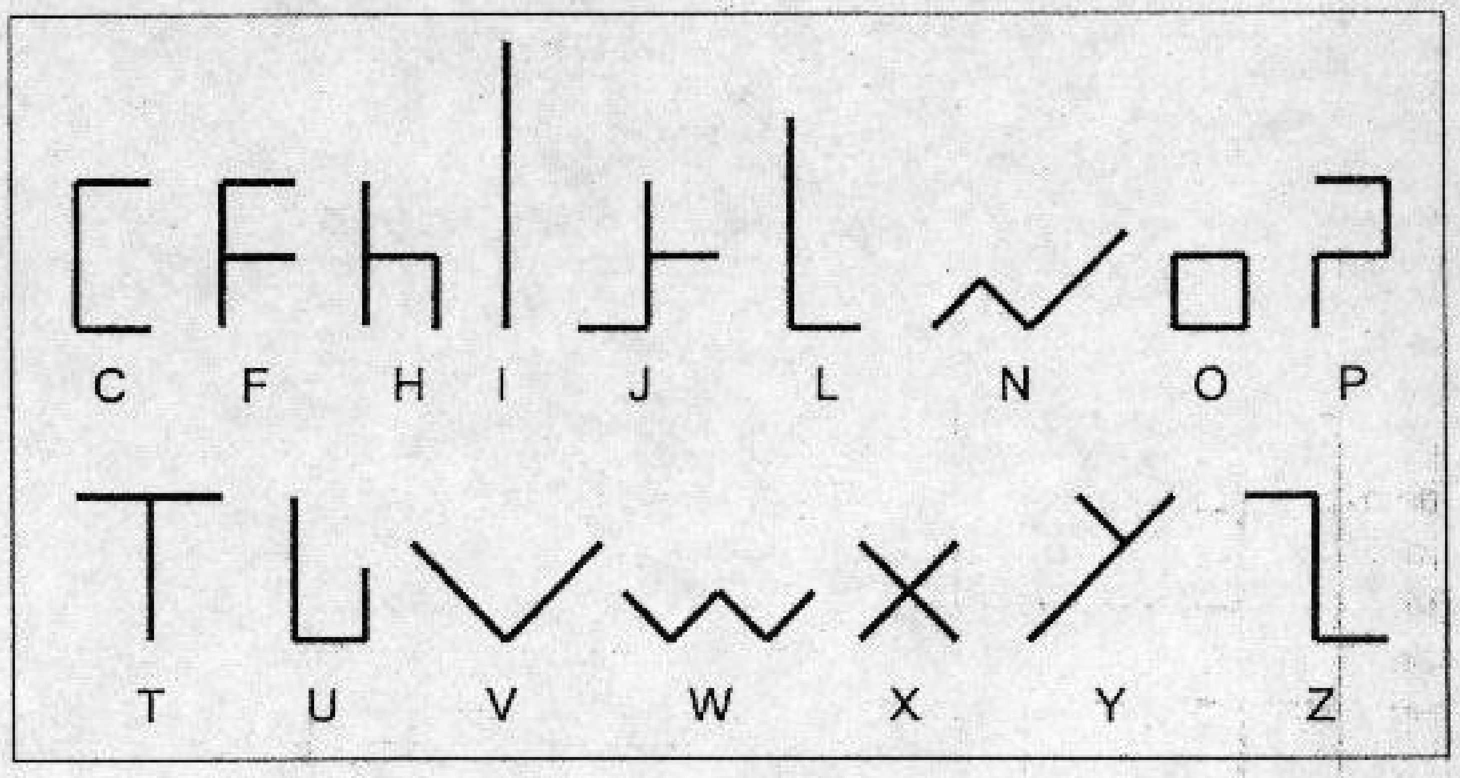

Figure 1. The sixteen tetrasticks.