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Third sound: Where are the solitons?

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With nanometer thicknesses and millimeter wavelengths, third sound is unquestionably in the long–wavelength limit of the classical fluid–dynamic description. The validity of the linearized form, successfully used for decades to predict the third sound speed, depends on the size of the dispersive terms in that description as well as on the amplitude. If the usual capillary dispersion is used, most third sound measurements would be far from linear and should consequently show a dramatically amplitude dependent speed. We report measurements of the frequency of a third sound resonator as a function of amplitude. Although there are uncertainties about the possible role of several dispersions, capillary dispersion appears to be negligibly small. A direct measure of the dispersion from the frequencies of higher resonant modes supports this.

1 INTRODUCTION

Third sound is a wave propagated in superfluid ^{4}He films analogous to long wavelength gravity waves in water. The Van der Waals attraction of the helium atoms to the substrate assumes the role of gravity. There is no question that third sound in these films is in the long wave limit of hydrodynamics. A typical wavelength is on the order of 1cm where the typical superfluid helium film thickness is atomic in scale – several nm. In fact, there are no other real systems which come close to this degree of shallowness without significant attenuation. As the amplitude of the wave disturbance increases, helium films should thus display nonlinear effects appropriate to this extreme limit.

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These phenomena have been studied for years in water, yet the sinusoidal, apparently linear behavior of third sound resonators seems to contradict these expectations.[1,2]

To explore this one must first look at the behavior of thin hydrodynamic films. Water waves are a natural place to start as the literature is extensive.[3] The small amplitude, long wave speed is $\sqrt{gh}$, where g is the constant acceleration of gravity and h is the film thickness. For larger amplitudes, a classic argument by Airy points out that since the wave speed near a crest is larger than the wave speed in a trough, the crests will eventually overtake the troughs and the leading edge of the crests will steepen until the wave breaks. It was subsequently shown that dispersive effects come to the rescue and allow steady waves without change of form. The dispersion comes from the breakdown of the shallow water assumption since steep waves imply short wavelengths. The resulting steady waves are the cnoidal solutions characteristic of the Korteweg–de Vries equation. These solutions at long wavelengths are a periodic sequence of humps with widths determined by the balance between the dispersive terms and the Airy steepening. The hump widths become independent of wavelength, effectively a series of solitons. Thus, even the long wavelength steady solutions involve short wavelength components.

How should third sound waves be different? First of all, because of the form of the Van der Waals potential, the effective gravity is a strongly decreasing function of the thickness h. In addition to introducing new non linear terms into the equations of motion, this makes the wave speed (again roughly $\sqrt{gh}$, proportional to $h^{-3/2}$) smaller for thicker films – opposite to that in water waves. Second, there are other dispersive terms in the equations of motion which can no longer be neglected. These can be cast in the form

$$C^2 = ghk^2(1+Ak^2) \qquad (1)$$

to make comparison among terms simpler. The most important term in long water waves is due to a correction for the vertical acceleration of fluid resulting in $A = -\frac{1}{3}h^2$. The generally shorter wavelengths typical of third sound make surface tension equally important, giving $A = \gamma/\rho g$, with γ the surface tension and ρ the fluid density. These are the two dispersions usually considered in treatments of third sound. Others will be discussed in section 3.

Given a generic dispersion as above, simplified equations of motion can be used to get a feel for the way the dispersion and nonlinearities keep each other in check. Only terms to second order in the amplitudes and lowest order in the dispersion are included. The height dependent Van der Waals acceleration is linearized as $g(\eta)=g_0(1+\alpha\frac{\eta}{h})$ with $\alpha=-4$. Simple solutions are possible in the extreme limits using standard techniques[3]. When $\eta/h>>Ak^2$, a soliton solution $\eta/h = H_0\text{sech}^2(k_0x)$ results with k_0 and speed given by

$$k_0^2 = -\frac{3+\alpha}{12A}H_0 \quad \text{and} \quad C^2 = gh(1-A|\frac{3+\alpha}{3A}H_0|) \quad . \qquad (2)$$

Note that the orientation of the hump must be chosen so that k_0^2 is positive and that the speed depends on the sign, but not the magnitude of the dispersion term. Careful consideration of dispersion – wave breaking balance backs this up: the speed is always determined by the linear wave speed evaluated for the hump's depth or elevation and must be opposed to the dispersive change for the shorter wavelength components of the hump. If $\eta/h<<Ak^2$ then a Stokes expansion of the form $\eta_1\cos(kx-\omega t)+\eta_2\cos(2kx-2\omega t)+...$ leads to

$$\eta_2=-\frac{3+\alpha}{12Ak^2}(\eta_1/h)^2 \qquad \text{and}$$

$$C^2=gh\{1-\frac{(3+\alpha)^2}{24Ak^2}(\eta_1/h)^2\} \ . \qquad (3)$$

The illuminating point here is that the near resonant relation of the second order mode to the non–linear terms applied to the first order mode is responsible for the disturbing reciprocal terms of Ak^2. The dispersion guarantees that $\omega(2k)\neq2\omega(k)$. If it were absent, this expansion, and hence the linear regime altogether, would not exist. It must be remembered that $\eta/h<<1$ in all cases: the soliton form does not liberate one to apply the solutions to large amplitudes.

2 EXPERIMENTS

There are two factors which determine whether, when all terms are properly accounted for in the equations of motion, long wave cnoidal type solutions will be experimentally relevant. Most important are the relative sizes of the nonlinear terms compared to the dispersive terms as seen above. The second factor will help clear up the apparent contradiction in linearized, non dispersive shallow waves. Supposing there was no dispersion, how long would it take for the second order terms to grow? This can be estimated by considering the time it would take for the fast part of the wave to catch up to the slower part. In the case of third sound, if the trough is faster than the crest by ΔC_3 then this time will be roughly when $\Delta C_3\tau=\lambda/2$. But for third sound, $\Delta C_3/C_3=-(\frac{3}{2}\Delta h/h)$ so that the steepening time is $\tau=\frac{\lambda}{3C_3}(\eta/h)^{-1}$. This can be made as long as one wishes with a small enough amplitude. Experimentally this time, and hence minimum amplitude is limited by either an observation time or decay of the wave. The latter is what makes the capillary range of shallow water waves inaccessible. Thin films of normal liquids are heavily damped.

2.1 Travelling Waves

Third sound pulses would seem to be ideal for soliton observation. Amplitudes of several percent of the film thickness are easily produced. Given the expected capillary dispersion and mm wavelengths, Ak^2 is about 10^{-11} . Third sound pulses are incredibly many orders of magnitude into the soliton regime, yet only hints of non linear components have been observed. Consider the time constraints stated above. An amplitude of six percent of the film thickness results in a soliton separation in about 100 widths of the pulse, much farther than typical detection distances. The suggested fix for this is to generate pulses with higher frequency components which results in a soliton train to be observed as an averaged disturbance. Spatially, this is no problem because of the relatively sharp edges of drive strips, but temporally the production of "square well" initial conditions is ruined by the diffusive nature of the substrate drives from which the films receive their energy. This temporal problem is particularly bad for the expected "cold", slower solitons as they trail the pulse and depend on the sharpness of the trailing edge of the heat applied to the film. This is certainly prolonged by the substrate diffusion.

The ideal amplitude for generating small numbers, or perhaps one soliton at a more reasonable .1mm width (see eq. 2) would require amplitudes with $\eta/h=10^{-4}$ to 10^{-5}. These amplitudes are possible to observe but the separation distance becomes more than 100 time longer than typical third sound plates. Accurate time of flight measurements of the small velocity shifts would conversely be extremely difficult on standard sized plates.

2.2 Resonators

The need to measure these small velocity shifts motivated the use of a third sound resonator. The resonator can be thought of as a long flight path folded up on itself. The limit on separation time is in this case intrinsic to the film given by the transient decay time of the wave $Q\lambda/\pi C_3$ with Q the quality factor. Thus one can also conclude that nonlinear effects will be observable if $\eta/h>1/Q$. Since Q's fall in the range of 10^4 to 10^6 a the resonator is useful for detecting the non linear effects over a wide range of amplitudes. Unlike pulsed time of flight experiments, a resonator is extremely sensitive to the small velocity shifts and should be useful for the detection of non linearity even at the lowest amplitudes.

It is important to first address the applicability of travelling wave solutions similar to those discussed above to a resonator. Korteweg DeVries solitons are able to pass through each other unchanged (this surprising feature appears to be a general property of a large class of non linear systems[4]) suggesting that an appropriate resonant soliton solution would be equivalent to two counter propagating trains of solitons truncated at the boundary of the resonator where the solitons pass through each other. This qualitative behavior is easily observed in the resonances of 1–2 cm of water in a square cake pan and we have successfully tested this quantitatively where the hydrodynamics for the travelling waves have been confirmed.

The frequency vs. amplitude information is obtained during third sound resonance free decays. The circular resonator is formed in the gap betwwn two microscope slides epoxied together with an 8.6μm gap. The flat surfaces are coated with evaporated silver partitioned into electrodes used for driving and detecting the helium thickness oscillations of the third sound modes electrostatically. The resonant modes are Bessel functions and the sensitivity of the pickup capacitor to the various modes is easily calculated from its geometry. Results are presented in terms of the mode amplitude η_m defined such that the film thickness oscillations are assumed to be

$$\Delta h(r,\phi,t) = \eta J_m(x_m r/a)\cos(m\phi)\cos(\omega t) \quad . \qquad (4)$$

Only the lowest angular modes are considered, designated by m, and x_m=1.841, 3.054... divided by the radius of the resonator, a, determine the wave number k.

A measurement is made by first driving the resonance up with an A.C. voltage of up to 20Vpp. The drive is then turned off and the amplitude and phase of the film oscillation are followed through the decay using a lock–in amplifier whose reference is fixed near the previously determined low amplitude resonant frequency. The instantaneous resonant frequency is then calculated from the rate of phase change. Fig.1 shows decay data plotted as the frequency shift vs. a quadratic mode amplitude scale. The scales have been normalized to the resonant frequency and the static film thickness, in this case 212.03 Hz and 4.5nm respectively. The straight line indicates a frequency shift proportional to square of the mode amplitude. Results for the slopes b of all of the data taken were described within the scatter by the empirical relation

$$b = \frac{C_3}{C_b}\left(1 + \left(\frac{k_b}{k}\right)^2\right) \qquad (5)$$

with C_b=379m/s and k_b=616m^{-1}. The third sound speed ranged from 9.2m/s to 25m/s and at least the three lowest modes are included in all but the highest third sound speeds where the measurements are very difficult.

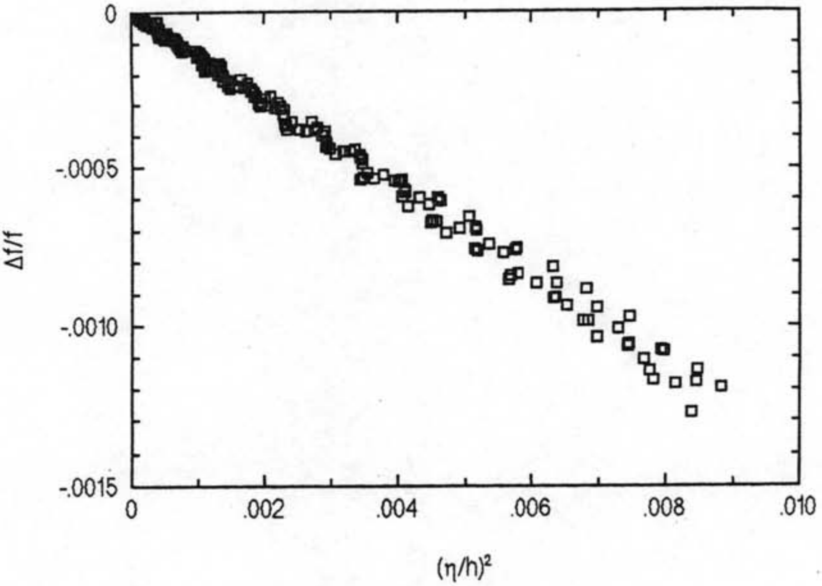

Fig. 1
The relative frequency shift vs. amplitude for a 4.5nm film at .05K . Note the quadratic amplitude scale.

3 DISPERSION

The results of the resonator shifts, when compared to either of the forms (2) or (3) of the velocity are inconsistent with the expected size of the non linearities. Since the resonator's transducers are completely electrostatic, there is no uncertainty about the amplitudes as there is in bolometer third sound pick-ups. We can turn this around and use the measured shifts to estimate the size of Ak^2 from the slopes (5) These results are represented in Fig. 2 as curve (a) along with several other estimates of Ak^2 which represent the transitions from $\eta/h << Ak^2$ to $\eta/h >> Ak^2$. To the left of any particular curve is the linear region and to the right is the soliton region. For all curves, k is chosen to be $k_1=300m^{-1}$, the wave number of the lowest mode in the resonator and the + or − sign refers to the sign of the dispersion. The range of amplitudes and film thicknesses investigated is also shown by the bracket above the amplitude axis.

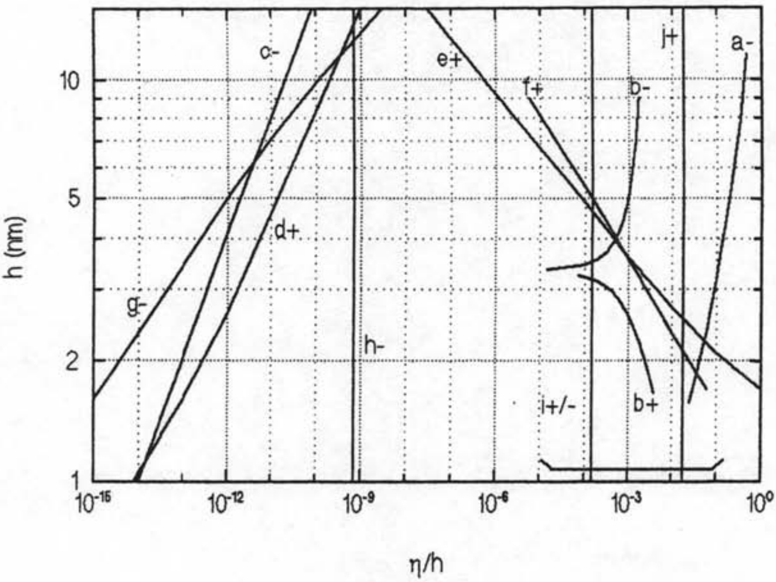

Fig. 2
Estimates of various dispersions are represented as curves showing where the cross–over from linear (to the left) to nonlinear (to the right) behavior would be. The curve labels are referred to in the text. The bracket shows typical third sound amplitudes

This result suggests that the dispersion in the resonator is many orders of magnitude greater than and of opposite sign to the capillary dispersion. We also measured the dispersion directly through the frequencies of the higher modes up to the (11,1) mode. These results are shown in Fig. 3 and plotted as curve (b) in Fig. 2. The number of modes measured drops down to two at the high third sound speeds and three at the low contributing to large errors at these ends. Note that the sign of this more direct measure of dispersion is opposite to that indicated by the frequency shifts, but still many orders of magnitude above the capillary dispersion.

What could be responsible for these contradictions? Other possible sources of dispersions will now be discussed to help answer this. The long wave (c) and capillarity (d) dispersions have already been presented. Two thermomechanical dispersions of the form

$$C^2 = C_{30}^2+C_5^2/(1+(\omega_0/\omega)^2) \quad (7)$$

have their roots in the thermal relaxation rate ω_0 of the film to the substrate or gas. In this expression, C_5 is the fifth sound speed. Curve (e) is for a film at 1.5K calculated by Bergman[5]. Curve (f) is based on the assumption of $\omega_0/\omega<<1$ from which the relaxation and thus the dispersion can be cast in terms of the measured Q.

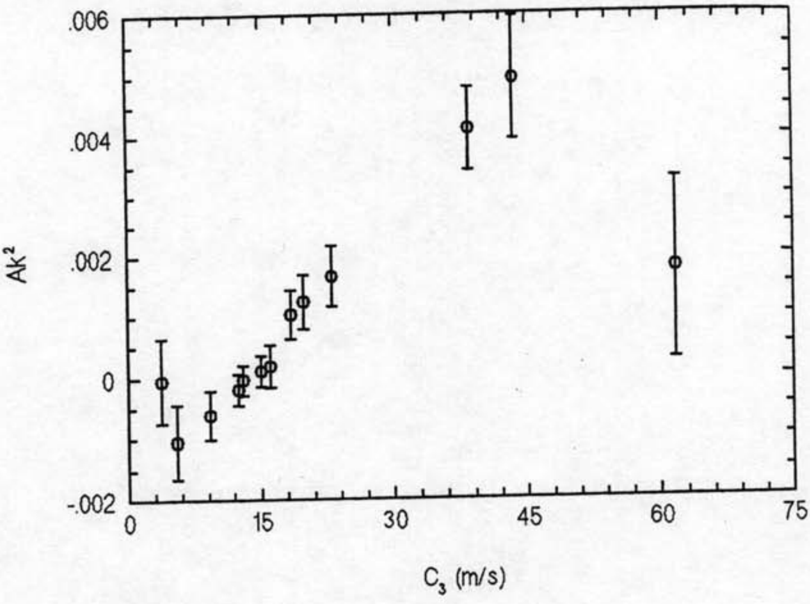

Fig. 3
Direct measure of the dispersion from the frequencies of up to the lowest 11 modes. The error bars indicate the uncertainty in the dispersion due to imperfections in the resonator and weak modes.

The last forms of dispersion are geometrical in nature. If there were a periodic modulation of the properties of the film then a gap and associated dispersion would be present. Surface roughness will have qualitatively the same effect – for wavelengths much longer than the characteristic length scale of the roughness, the similarity should be even better. The size of the dispersion can thus be estimated as[6]

$$Ak^2 = -(\Delta z/z)^2(k/k_0)^2 \quad (8)$$

here $\Delta z/z$ is the relative modulation of the third sound mechanical impedance and k_0 characterizes the roughness length scale. Two curves are shown for this case, (g) for the expected speed modulation for films adsorbed on a surface with 3nm roughness amplitude and a 100nm length scale typical of evaporated and annealed gold surfaces. The curve (h) is an estimate for dirt particles with 100μm separation and a $\Delta z/z$ due to the local adsorption of film volume by capillarity.

Finally, there is an effective dispersion which a resonator would have if the modes are not harmonically related. Recall that the key role of the dispersion was to shift the mode at 2k away from resonance at 2ω. Anything that shifts the modes away from this relationship has the same effect. This "geometrical dispersion" could be due to imperfections in the resonator or just simply the geometry of the resonator. For comparison, one can find the dispersion that would have the same effect as the imperfection with

$$Ak^2 = \frac{1}{6}(k_2/k_1-2) \qquad (9)$$

where k_1 and k_2 are the wave numbers of the near-resonant modes. Imperfections are always present. In a series of 5 nearly identical resonators, what should be degenerate modes were always split, typically less than one part per thousand. This "dispersion" shown as curve (i) could be of either sign.

By far the largest in the present case of the circular resonator is simply the geometry of the Bessel modes. They are not harmonically related. Curve (j) is for the (1,1) mode whose non linear terms are nearly resonant with the (0,1) mode. These geometrical dispersions would appear to make all resonators, especially non rectangular ones, insensitive to intrinsic dispersions in the films. On the other hand, these same imperfections will plague the travelling waves with substrate defects or deviations from perfect plane waves in the case of third sound pulses generated by drive strips of finite extent. We are presently investigating the validity of these last points.

4 CONCLUSIONS

We have speculated about several possible dispersions which should be taken into account for a realistic treatment of soliton propagation in superfluid helium films. The ideal dispersions associated with surface tension and vertical acceleration, due to their extremely small size, are most likely swamped by thermomechanical or geometric dispersions. If this is true, helium films may not be as nonlinear as previously thought and may explain the conspicuously absent nonlinear behavior expected with only capillary dispersion. Experiments in a rectangular resonator are planned, but imperfections will be difficult to distinguish from dispersions.

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