Third Sound Equations of Motion (MKS units)

Last checked and corrected -- 07/20/09

This worksheet decsribes the equations of motion along the lines of Bergman's original paper with a few simplifications appropriate for low temperatures (roughly T<0.5): (1) The normal component velocity is set to zero; (2) The substrate and vapor are taken as thermal reservoirs with no internal dynamics; and (3) The vapor interactions are taken to be purely ballistic with a 100% accomodation coefficient. An additional term demonstrating the gain associated with stimulated condensation is also included.

See Bergman's papers for the complete adaption of the two fluid model to third sound:

"Hydrodynamics and Third Sound in Thin HeII Films
D. Bergman, Phys. Rev. **188**, 370 (1969)

"Third Sound in Superfluid Films of Arbitrary Thickness",
D. J. Bergman, Phys. Rev. **A3**, 2058 (1971)

▼ Physical Constants

$\hbar := 1.05457168 \cdot 10^{-34}$ $m := 6.646 \cdot 10^{-27}$ $k := 1.380650510^{-23}$

substrate van der Waals strength $T_v := 40$

liquid density $\rho := 145.1397$ monolayer thickness $h_1 := \left(\frac{m}{\rho}\right)^{\frac{1}{3}}$

latent heat of vaporization $L_0 := 7.196$

▲ Physical Constants

Variables and Parameters Used

h	static film thickness
η	surface displacement
ρ	bulk helium density
ρ_s	effective ground state mass density on the film
v	superfluid film flow velocity
J_m	mass flux into film from vapor
g	van der Waals acceleration of a free atom at the height of the surface

$T \quad \tau \quad S \quad C$	film temperature, temperature oscillations, entropy, heat cap per mass
$\bar{S}$	partial entropy per mass $\bar{S} = \frac{\delta}{\delta h}(h \cdot S)$
L	latent heat per mass
K	film to substrate thermal contact parameter (Kapitza conductance)
σ	lateral conductance
Φ	ideal gas equilibrium particle flux
Φ_{net}	net particle flux due to stimulated condensation vapor source

Minimal Third Sound

The essence of third sound is identical to long-length ocean waves (tidal waves). The basic equations of motion express conservation of mass and Newton's law. Formally they come from basic fluid mechanics...

(1) incompressible flow $\nabla \cdot \mathbf{v} = 0$

Assume that the velocity in the z direction ranges linearly from 0 at the substrate to $\frac{\delta \eta}{\delta x}$ at the free surface, z=h:

$$\mathbf{v} = \mathbf{v_{xy}} + \mathbf{k} \frac{z}{h} \cdot \frac{\delta \eta}{\delta t}$$

With no y dependence, $\nabla \cdot \mathbf{v} = 0$ gives

$$\frac{d}{dt}\eta = -h \cdot \nabla \cdot \mathbf{v_{xy}} \qquad \text{conservation of mass}$$

This expression for the incompressibility of the fluid shows that fluid diverging drom a region reduces the film thickness.

(2) Euler's equation with a fixed force per mass g $\frac{\delta \mathbf{v}}{\delta t} = -\frac{1}{\rho} \cdot \nabla P - g$

Assume that the pressure has a hydrostatic z dependence and a small lateral dependence...

$$P = \rho \cdot g \cdot (h - z) + P_{xy}$$

This satisfies the static (g) term, and requiring that P=0 at the displaced surface z=h+η identifies the lateral dependence as $P_{xy} = \rho \cdot g \cdot \eta$. Euler's equation, again with no y dependence, gives...

$$\frac{d}{dt}v = -g \cdot \nabla \eta \qquad \text{Euler's equation}$$

Fluid accelerates from higher pressure below the deeper regions toward the lower pressure under shallower regions.

The minimal third sound equations in 1D (the x direction) are then

$$\frac{d}{dt}\eta = -h \cdot \nabla \cdot \mathbf{v_{xy}} \qquad \frac{d}{dt}\mathbf{v_{xy}} = -g \cdot \nabla \eta$$

Eliminating the velocity gives the wave equation for the surface displacements η and identifies the third sound speed...

$$\nabla^2 \eta = \frac{1}{c^2} \cdot \frac{d^2}{dt^2}\eta \qquad c_3{}^2 = g \cdot h$$

More Complete Derivation

Two features will be added to the simple model above: (1) In the context of the "two fluid model", the normal component hydrodynamics are ignored. However, any small diffusive motion of the normal component and thermal conversion between heat and the normal component will be accounted for by the flow of heat. (2) The presence of the vapor will be included. The modifications to the above relations and a new relation for heat flow are explained below:

Conservation of Mass Film thicknedd changes defined by the total fluid density are brought about through flow at the superfluid density or addition of new particled from the vapor. J_m is mass flux into the film from the vapor.

$$\rho \cdot \frac{d}{dt}\eta = -h \cdot \rho_s \cdot \frac{d}{dx}v + J_m$$

Euler's Equation The pressure gradient imposed by film thickness differences is replaced by the chemical potential differences per mass: $d\mu = \frac{1}{\rho} \cdot dP - S \cdot dT$. The entropy, however, needs to be adjusted for the case of the film with the free surface, essentially a new thermodynamic variable. The "partial entropy" defined by Bergman invokes a thermomechanical force only for that part of the entropy that can be diluted by superflow, i.e., only proportional to how much the total entropy per area changes with thickness. The stimulated condensation gain term is the net force per mass associated with the momentum of particle accomodation into the ground state fluid.

$$\frac{F}{M} = \frac{1}{M}\cdot\frac{\Delta p}{\Delta t} = \frac{m\cdot\Phi_{net}\cdot A\cdot v}{\rho\cdot h\cdot A}$$

$$\frac{d}{dt}v = -g\cdot\frac{d}{dx}\eta + \bar{S}\cdot\frac{d}{dx}\tau + \frac{m\cdot\Phi_{net}}{\rho\cdot h}\cdot v \qquad \bar{S} = \frac{1}{\rho}\cdot\left(\frac{\delta\Sigma}{\delta h}\right)_T = \frac{\delta}{\delta h}(h\cdot S) \quad \text{fixed T}$$

Heading: Heat Flow The total entropy per area (Σ) changes only by heat flow including (1) latent heat from net condensing gas particles, (2) temperature imbalance of the exchanging vapor particles, (3) Thermal exchange with the substrate, and (4) lateral conduction within the film...

$$\frac{d}{dt}\Sigma = \frac{J_Q}{T}$$

Use... $\Sigma = \rho\cdot h\cdot S$ and $J_Q = L\cdot J_m - \frac{3}{2}\cdot k\cdot\Phi\cdot\tau - K\cdot\tau + \sigma\cdot\frac{d^2}{dx^2}\tau$

The first is the entropy per area. The second defines the four heat sources mentioned above. With careful differentiation of Σ (illustrating the origin of the partial entropy in a simpler context than above)...

$$\frac{d\Sigma}{dt} = \frac{\delta\Sigma}{\delta T}\cdot\frac{dT}{dt} + \frac{\delta\Sigma}{\delta h}\cdot\frac{d\eta}{dt} \qquad \left(\frac{\delta\Sigma}{\delta h}\right)_T = \rho\cdot\bar{S} \qquad \left(\frac{\delta\Sigma}{\delta T}\right)_h = \frac{\rho\cdot h\cdot C}{T}\cdot\frac{dT}{dt}$$

Combine all this into...

$$\rho\cdot h\cdot C\cdot\frac{d\tau}{dt} + \rho\cdot T\cdot\bar{S}\cdot\frac{d\eta}{dt} = L\cdot J_m - \frac{3}{2}\cdot k\cdot\Phi\cdot\tau - K\cdot\tau + \sigma\cdot\frac{d^2}{dx^2}\tau$$

▼ Vapor Properties

The interaction with the vapor is assumed to be ballistic. The ideal gas particle flux Φ through a surface is assumed to be the flux into and out of the film's surface in equilibrium. With the gas properties fixed at this condition, the detailed balance is upset by either a temperature difference or a thickness difference of the film away from equilibrium. The return flux from the film is assumed to that of the equilibruim vapor at the perturbed conditions.

The flux J_m is defined *into* the film. Changes in flux result from changes in the local film parameters h and T AND from the moving surface sweeping out vapor particles. Unless otherwise motivated, use the far equilibrium gas conditions as that at the free surface.

$$\Phi = \Phi(h,T) = \frac{P_{film}(h,T)}{\sqrt{2\cdot\pi\cdot m\cdot k\cdot T}}$$

$$\Delta\Phi = \Phi(h+\eta, T+\tau) - \Phi(h,T) = \frac{\delta\Phi}{\delta T}\cdot\tau + \frac{\delta\Phi}{\delta h}\cdot\eta$$

The mass flux is this imbalance PLUS an extra amount due to the motion of the film surface. An upward moving film surface intercepts an additional amount as it sweeps through the gas density.

$$J_m = -m\cdot\Delta\Phi + \rho_{gas}\cdot\frac{d\eta}{dt}$$ J into the film, ΔΦ net flux out of film $$\rho_{gas} = \frac{m\cdot P_{film}}{k\cdot T}$$

$$J_m = m\cdot\Phi\cdot\left[-\frac{1}{\Phi}\cdot\left(\frac{\delta\Phi}{\delta T}\cdot\tau + \frac{\delta\Phi}{\delta h}\cdot\eta\right) + \sqrt{\frac{2\cdot\pi\cdot m}{k\cdot T}}\cdot\frac{\delta\eta}{\delta t}\right]$$

Note that this mass flux represents the oscillatory component only. Φ_{net} couples to the oscillation equations only through it's accomodation into the local flow.

The vapor pressure relation below is valid for T < ~ 1 K and includes the Boltzman factor for the film.

unsaturated pressure $$P(h,T) := \left(\frac{m}{2\cdot\pi\cdot\hbar^2}\right)^{\frac{3}{2}}\cdot(k\cdot T)^{\frac{5}{2}}\cdot\exp\left[-\frac{L_0}{T} - \frac{T_v}{T}\cdot\left(\frac{h_1}{h}\right)^3\right]$$

particle flux from kinetic theory $$\Phi(h,T) := \frac{P(h,T)}{\sqrt{2\cdot\pi\cdot m\cdot k\cdot T}}$$

or $$\Phi(h,T) := \frac{m}{\hbar^3}\cdot\left(\frac{k\cdot T}{2\cdot\pi}\right)^2\cdot\exp\left[-\frac{L_0}{T} - \frac{T_v}{T}\cdot\left(\frac{h_1}{h}\right)^3\right]$$

The derivatives above can then be expressed interms of the unitless functions of T and h...

$$A_T = \frac{T}{\Phi}\cdot\frac{\delta\Phi}{\delta T} = 2 + \frac{L_0}{T} + \frac{T_v}{T}\cdot\left(\frac{h_1}{h}\right)^3 \qquad A_h = \frac{h}{\Phi}\cdot\frac{\delta\Phi}{\delta h} = 3\cdot\frac{T_v}{T}\cdot\left(\frac{h_1}{h}\right)^3$$

Vapor Properties

Summary Equations and Solution

incompressible $$\rho\cdot\frac{d}{dt}\eta = -h\cdot\rho_s\cdot\frac{d}{dx}v + J_m$$

Euler $$\frac{d}{dt}v = -g\cdot\frac{d}{dx}\eta + \bar{S}\cdot\frac{d}{dx}\tau + \frac{m\cdot\Phi_{net}}{\rho\cdot h}\cdot v$$

heat balance $$\rho\cdot h\cdot C\cdot\frac{d\tau}{dt} + \rho\cdot T\cdot\bar{S}\cdot\frac{d\eta}{dt} = L\cdot J_m - \frac{3}{2}\cdot k\cdot\Phi\cdot\tau - K\cdot\tau + \sigma\cdot\frac{d^2}{dx^2}\tau$$

vapor exchange $$J_m = m\cdot\Phi\cdot\left[-\frac{1}{\Phi}\cdot\left(\frac{\delta\Phi}{\delta T}\cdot\tau + \frac{\delta\Phi}{\delta h}\cdot\eta\right) + \sqrt{\frac{2\cdot\pi\cdot m}{k\cdot T}}\cdot\frac{\delta\eta}{\delta t}\right]$$

Cast into matrix form with unitless dynamical variables $\left(\frac{\eta}{h},\frac{v}{c},\frac{\tau}{T}\right)$ oscillating as $e^{i\cdot(q\cdot x-\omega\cdot t)}$ ...

$$\begin{bmatrix} m\cdot\Phi\cdot\left(i\cdot\frac{\omega}{\gamma_h} + A_h\right) - i\cdot\omega\cdot\rho\cdot h & i\cdot q\cdot\rho_s\cdot h\cdot c & m\cdot\Phi\cdot A_T \\ i\cdot q\cdot g\cdot h & -i\cdot\omega\cdot c - \frac{m\cdot\Phi_{net}}{\rho\cdot h}\cdot c & -i\cdot q\cdot T\cdot\bar{S} \\ \frac{m\cdot\Phi\cdot L}{T}\cdot\left(i\cdot\frac{\omega}{\gamma_h} + A_h\right) - i\cdot\omega\cdot\rho\cdot h\cdot\bar{S} & 0 & \frac{m\cdot\Phi\cdot L}{T}\cdot A_T + \frac{3}{2}\cdot k\cdot\Phi + K + \sigma\cdot q^2 - i\cdot\omega\cdot\rho\cdot h\cdot C \end{bmatrix}\cdot\begin{pmatrix}\frac{\eta}{h}\\ \frac{v}{c}\\ \frac{\tau}{T}\end{pmatrix}$$

using the following definitions...

$$\gamma_h = \frac{1}{h}\cdot\sqrt{\frac{k\cdot T}{2\cdot\pi\cdot m}} \qquad c = \gamma_h\cdot h = \sqrt{\frac{k\cdot T}{2\cdot\pi\cdot m}}$$

$$A_T = \frac{T}{\Phi}\cdot\frac{\delta\Phi}{\delta T} = 2 + \frac{L_0}{T} + \frac{T_v}{T}\cdot\left(\frac{h_1}{h}\right)^3 \qquad A_h = \frac{h}{\Phi}\cdot\frac{\delta\Phi}{\delta h} = 3\cdot\frac{T_v}{T}\cdot\left(\frac{h_1}{h}\right)^3$$

Check the no-vapor (Φ, Φ_{net}=0) case for insight into the full solution...

$$\begin{vmatrix}\begin{pmatrix} -i\cdot\omega\cdot\rho & i\cdot q\cdot\rho_s\cdot c & 0 \\ i\cdot q\cdot g\cdot h & -i\cdot\omega\cdot c - \frac{m\cdot\Phi_{net}}{\rho\cdot h}\cdot c & -i\cdot q\cdot T\cdot\bar{S} \\ -i\cdot\omega\cdot\rho\cdot h\cdot\bar{S} & 0 & K + \sigma\cdot q^2 - i\cdot\omega\cdot\rho\cdot h\cdot C \end{pmatrix}\end{vmatrix}\cdot\begin{vmatrix}\begin{pmatrix}\frac{\eta}{h}\\ \frac{v}{c}\\ \frac{\tau}{T}\end{pmatrix}\end{vmatrix} = 0$$

This is most easily solved by solving for τ in the last row and rewriting the 2x2 determinant for the others and solving...

$$\frac{\tau}{T} = -\frac{\bar{S}}{C}\cdot\frac{1}{\left(1 + i\cdot\frac{K+\sigma\cdot q^2}{\omega\cdot\rho\cdot h\cdot C}\right)}\cdot\frac{\eta}{h} \qquad \omega^2 = q^2\cdot\frac{\rho_s}{\rho}\cdot g\cdot h\cdot\left[1 + \frac{T\cdot\left(\bar{S}\right)^2}{C\cdot g\cdot h}\cdot\frac{1}{\left(1 + i\cdot\frac{K+\sigma\cdot q^2}{\omega\cdot\rho\cdot h\cdot C}\right)}\right]$$

This is the low temperature limit where only non-vapor conduction is important. Two further limits are...

isothermal - perfect conduction $\frac{\tau}{T} = 0$ $\omega^2 = q^2 \cdot \frac{\rho_s}{\rho} \cdot g \cdot h$

adiabatic - no conduction $\frac{\tau}{T} = -\frac{\overline{S}}{C} \cdot \frac{\eta}{h}$ $\omega^2 = q^2 \cdot \frac{\rho_s}{\rho} \cdot g \cdot h \cdot \left[1 + \frac{T \cdot \left(\overline{S}\right)^2}{C \cdot g \cdot h}\right]$

Note that the importance of the thermomechanical terms is gauged by essentiallt the ratio $\frac{T \cdot \left(\overline{S}\right)^2}{C \cdot g \cdot h}$

Motivated by the same limit, follow the same manipulations with some new definitions......

$\gamma_h = \frac{c}{h} = \frac{1}{h} \cdot \sqrt{\frac{k \cdot T}{2 \cdot \pi \cdot m}}$ atomic speed in film thicknesses per time

$\gamma_v = \frac{k \cdot \Phi}{\rho \cdot h \cdot C}$ film-vapor thermal exchange rate

$\gamma_s = \frac{K + \sigma \cdot q^2}{\rho \cdot h \cdot C}$ film-substrate thermal exchange rate

$\gamma_{netx} = \frac{m \cdot \Phi_{net}}{\rho \cdot h}$ net film particle exchange rate

Solve for τ/T in the last row
of the 3x3

$$\frac{\tau}{T} = -\frac{\frac{\overline{S}}{C} + i \cdot \frac{\gamma_v}{\omega} \cdot \frac{L_0}{T} \cdot \left(i \cdot \frac{\omega}{\gamma_h} + A_h\right)}{1 + i \cdot \frac{\gamma_v}{\omega} \cdot \left(\frac{L_0}{T} \cdot A_T + \frac{3}{2}\right) + i \cdot \frac{\gamma_s}{\omega}} \cdot \frac{\eta}{h} = -D \cdot \frac{\eta}{h} \qquad L = \frac{k \cdot L_0}{m}$$

Use in the remainig
2x2 system

$$\begin{bmatrix} m \cdot \Phi \cdot \left(i \cdot \frac{\omega}{\gamma_h} + A_h\right) - i \cdot \omega \cdot \rho \cdot h - m \cdot \Phi \cdot A_T \cdot D & i \cdot q \cdot \rho_s \cdot h \cdot c \\ i \cdot q \cdot g \cdot h + i \cdot q \cdot T \cdot \overline{S} \cdot D & -i \cdot \omega \cdot c - \gamma_{netx} \cdot c \end{bmatrix} \cdot \begin{pmatrix} \frac{\eta}{h} \\ \frac{v}{c} \end{pmatrix} = 0$$

$$v = \frac{q}{\omega}\cdot\frac{\left(g\cdot h + T\cdot\bar{S}\cdot D\right)}{1 - i\cdot\frac{m\cdot\Phi_{net}}{\omega\cdot\rho\cdot h}}\cdot\frac{\eta}{h}$$

$$\omega^2 = q^2\cdot\frac{\rho_s}{\rho}\cdot\left(g\cdot h + T\cdot\bar{S}\cdot D\right) - i\cdot\omega\cdot\frac{m\cdot C\cdot\gamma_v}{k}\cdot\left(i\cdot\frac{\omega}{\gamma_h} + A_h - A_T\cdot D\right) + \gamma_{netx}\cdot\left[i\cdot\omega - \frac{m\cdot C\cdot\gamma_v}{k}\cdot\left(i\cdot\frac{\omega}{\gamma_h} + A_h - A_T\cdot D\right)\right]$$

$$D = \frac{\frac{\bar{S}}{C} + i\cdot\frac{\gamma_v}{\omega}\cdot\frac{L_0}{T}\cdot\left(i\cdot\frac{\omega}{\gamma_h} + A_h\right)}{1 + i\cdot\frac{\gamma_v}{\omega}\cdot\left(\frac{L_0}{T}\cdot A_T + \frac{3}{2}\right) + i\cdot\frac{\gamma_s}{\omega}}$$

Note that D is the ratio of temperature oscillations to thickness oscillations and cannot be greater in magnitude than the adiabic limit.

Vapor Exchange Only $K = \sigma = \Phi_{net} = 0$

$$D = \frac{\frac{\bar{S}}{C} + i\cdot\frac{\gamma_v}{\omega}\cdot\frac{L_0}{T}\cdot\left(i\cdot\frac{\omega}{\gamma_h} + A_h\right)}{1 + i\cdot\frac{\gamma_v}{\omega}\cdot\left(\frac{L_0}{T}\cdot A_T + \frac{3}{2}\right)}$$

$$\omega^2 = q^2\cdot\frac{\rho_s}{\rho}\cdot\left(g\cdot h + T\cdot\bar{S}\cdot D\right) - i\cdot\omega\cdot\frac{m\cdot C\cdot\gamma_v}{k}\cdot\left(i\cdot\frac{\omega}{\gamma_h} + A_h - A_T\cdot D\right)$$

$$D = \frac{\frac{L_0}{T}\cdot\left[3\cdot\frac{T_v}{T}\cdot\left(\frac{h_1}{h}\right)^3\right]}{\left[\frac{L_0}{T}\cdot\left[2 + \frac{L_0}{T} + \frac{T_v}{T}\cdot\left(\frac{h_1}{h}\right)^3\right] + \frac{3}{2}\right]} = \frac{1}{1 + \frac{1}{3}\cdot\frac{L_0}{T_v}\cdot\left(\frac{h}{h_1}\right)^3}$$

=1/10 approximately

$$\omega^2 = q^2\cdot\frac{\rho_s}{\rho}\cdot\left(g\cdot h + T\cdot\bar{S}\cdot D\right) - i\cdot\omega\cdot\frac{m\cdot C\cdot\gamma_v}{k}\cdot\left(i\cdot\frac{\omega}{\gamma_h} + A_h - A_T\cdot D\right)$$

$$A_T = \frac{T}{\Phi}\cdot\frac{\delta\Phi}{\delta T} = 2 + \frac{L_0}{T} + \frac{T_v}{T}\cdot\left(\frac{h_1}{h}\right)^3 \qquad A_h = \frac{h}{\Phi}\cdot\frac{\delta\Phi}{\delta h} = 3\cdot\frac{T_v}{T}\cdot\left(\frac{h_1}{h}\right)^3$$

...looking for $1 + \frac{T\cdot S}{L}$ terms

Stimulated Condensation

The stimulated condensation gain compensates for damping. Assuming it and the damping terms are small, the solution is basically, with representing all of the loss trms (approximately)

$$\omega^2 = \omega_0{}^2 + i\cdot\omega\cdot\left(\gamma_{netx} - \gamma_{loss}\right) - \frac{m\cdot C\cdot\gamma_v}{k}\cdot\left(i\cdot\frac{\omega}{\gamma_h} + A_h - A_T\cdot D\right)\cdot\gamma_{netx} \qquad \gamma_{netx} = \frac{m\cdot\Phi_{net}}{\rho\cdot h}$$

The last term proportional to γ_{netx} is usually small, but could be as high as 10% at the higher temperatures. It will slightly decrease the gain and shift the frequency down relative to the first iω term alone.

The effect on the Q of the third sound (neglecting the small correction) is

$$\frac{1}{Q} = \frac{1}{Q_0} - \frac{m\cdot\Phi_{net}}{\omega_0\cdot\rho\cdot h}$$

The critical flux for self oscillation is

$$\Phi_{crit} = \frac{\omega_0\cdot\rho\cdot h}{m\cdot Q_0} \qquad \text{or} \qquad \frac{dN}{dt} = 2\cdot\frac{N_{film}}{\tau_{decay}}$$

which states that all the atoms in the film need to be replaced in half the free decay time of the third sound.

Numerical Evaluation

(2,1)S mode wavenumber of the stimulated condensation resonator $\qquad q := \frac{3.05424}{0.00675} \qquad q = 452.48$

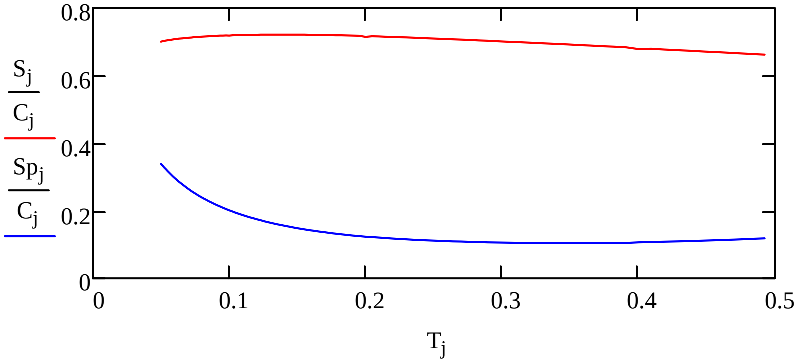
Pick up the thermodynamic data from "Film Thermodynamics a.xls" after taking the partial entropy derivative. The resuls are in "numerical thermo results.xls" for a few thicknesses.

`dat := READPRN("b:\physics\films\thermal\sim_dat\thermo_3.txt")` $\qquad h := 3\cdot 10^{-9}$

$j := 0 .. \mathrm{rows}(dat) - 1 \quad T_j := dat_{j,0}$ $\qquad c30 := \sqrt{\frac{3\cdot k\cdot T_v}{m}\cdot\left(\frac{h_1}{h}\right)^3} \qquad g := \frac{c30}{h}$

$S_j := dat_{j,1} \qquad Sp_j := dat_{j,2} \qquad C_j := dat_{j,3} \qquad nf_j := 1 - 1.46\cdot\frac{h_1}{h} - dat_{j,4}$

Here is a plot of the entropy and partial entropy to heat capacity ratio



This is the atomic speed parameter $\qquad \gamma_{h_j} := \frac{1}{h}\cdot\sqrt{\frac{k\cdot T_j}{2\cdot\pi\cdot m}}$

This is the thermal vapor cooling rate $\qquad \gamma_{v_j} := \frac{k\cdot\Phi\left(h, T_j\right)}{\rho\cdot h\cdot C_j}$

Stimulated condensation film exchange rate -- the heater loop particle flux measured in film equivalents per second. $\qquad \gamma_{netx} := 0$

Subtrate thermal relaxation rate $\qquad \gamma_{s_j} := 1000$

Use the partial entropy $\qquad S_j := Sp_j$

vaper parameters $\qquad A_{T_j} := 2 + \frac{L_0}{T_j} + \frac{T_v}{T_j}\cdot\left(\frac{h_1}{h}\right)^3 \qquad A_{h_j} := 3\cdot\frac{T_v}{T_j}\cdot\left(\frac{h_1}{h}\right)^3$

This is the normallized negative ratio of thermal oscillations to thickness oscillations

$$\frac{\tau}{T} = -D\cdot\frac{\eta}{h}$$

$$D(j,\omega) := \frac{\dfrac{S_j}{C_j} + i\cdot\dfrac{\gamma_{v_j}\cdot L_0}{\omega\cdot T_j}\cdot\left(i\cdot\dfrac{\omega}{\gamma_{h_j}} + A_{h_j}\right)}{1 + i\cdot\dfrac{\gamma_{v_j}}{\omega}\cdot\left(\dfrac{L_0}{T_j}\cdot A_{T_j} + \dfrac{3}{2}\right) + i\cdot\dfrac{\gamma_{s_j}}{\omega}}$$

$$\omega^2 = q^2\cdot\frac{\rho_s}{\rho}\cdot\left(g\cdot h + T\cdot\bar{S}\cdot D\right) - i\cdot\omega\cdot\frac{m\cdot C\cdot\gamma_v}{k}\cdot\left(i\cdot\frac{\omega}{\gamma_h} + A_h - A_T\cdot D\right) + \gamma_{netx}\cdot\left[i\cdot\omega - \frac{m\cdot C\cdot\gamma_v}{k}\cdot\left(i\cdot\frac{\omega}{\gamma_h} + A_h - A_T\cdot D\right)\right]$$

Solve this for ω:

$$\Omega(j,\omega) := \left|\begin{aligned} & u \leftarrow \gamma_{v_j}\cdot\frac{m\cdot C_j}{k}\cdot\left(i\cdot\frac{\omega}{\gamma_{h_j}} + A_{h_j} - A_{T_j}\cdot D(j,\omega)\right) \\ & \sqrt{q^2\cdot nf_j\cdot\left(c30^2 + T_j\cdot S_j\cdot D(j,\omega)\right) + \gamma_{netx}\cdot(i\cdot\omega - u) - i\cdot\omega\cdot u} \end{aligned}\right.$$

Start at low T...

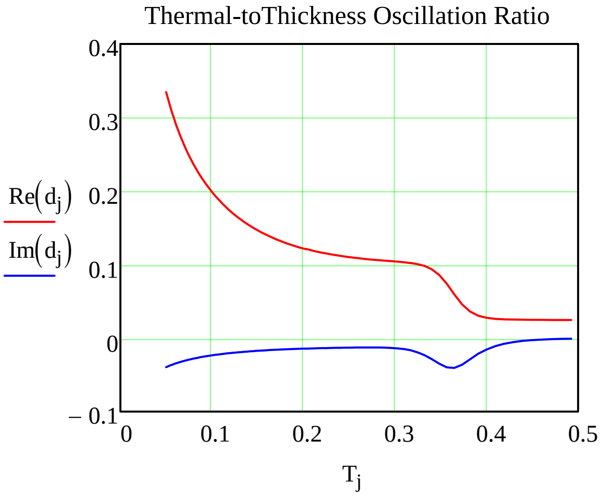
$$\omega0 := \sqrt{nf_0}\cdot c30\cdot q \qquad \omega_0 := \mathrm{root}(\Omega(0,\omega0) - \omega0, \omega0) \qquad \omega_0 = 8.455\times 10^3 - 5.438i\times 10^{-3}$$

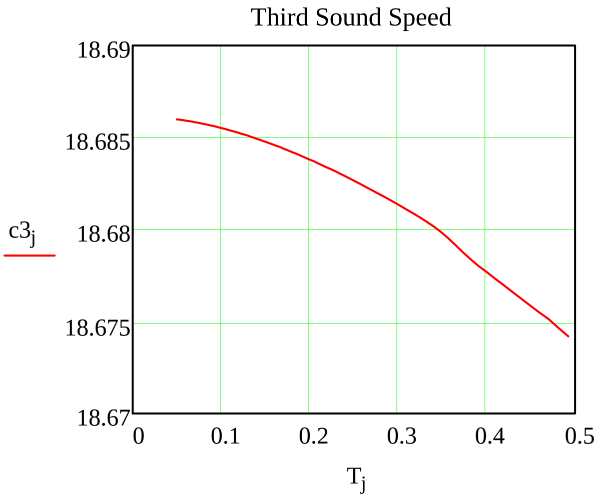
Step up in T using the previous solution for the root guess...

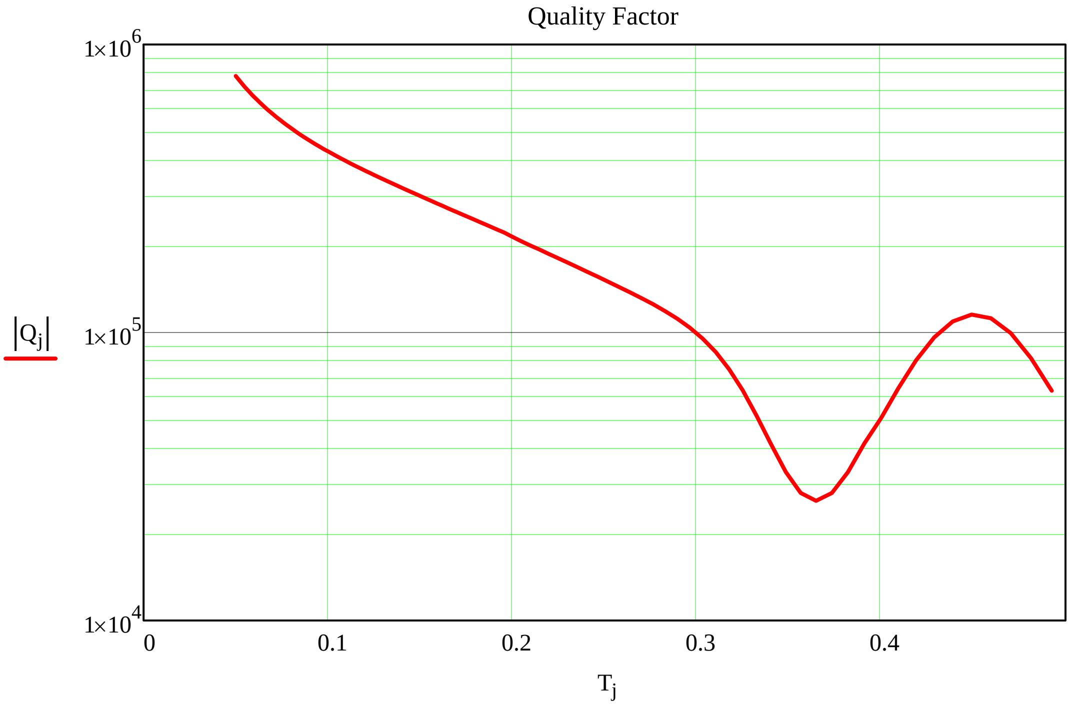
$$j := 1 .. \mathrm{rows}(dat) - 1 \qquad \omega_j := \left|\begin{aligned} & u \leftarrow \omega_{j-1} \\ & \mathrm{root}(\Omega(j,u) - u, u) \end{aligned}\right.$$

Calculate some other interesting numbers...

$$j := 0 .. \mathrm{rows}(dat) - 1 \quad d_j := D\left(j,\omega_j\right) \qquad c3_j := \frac{\mathrm{Re}\left(\omega_j\right)}{q} \qquad Q_j := -\frac{1}{2}\cdot\frac{\mathrm{Re}\left(\omega_j\right)}{\mathrm{Im}\left(\omega_j\right)}$$

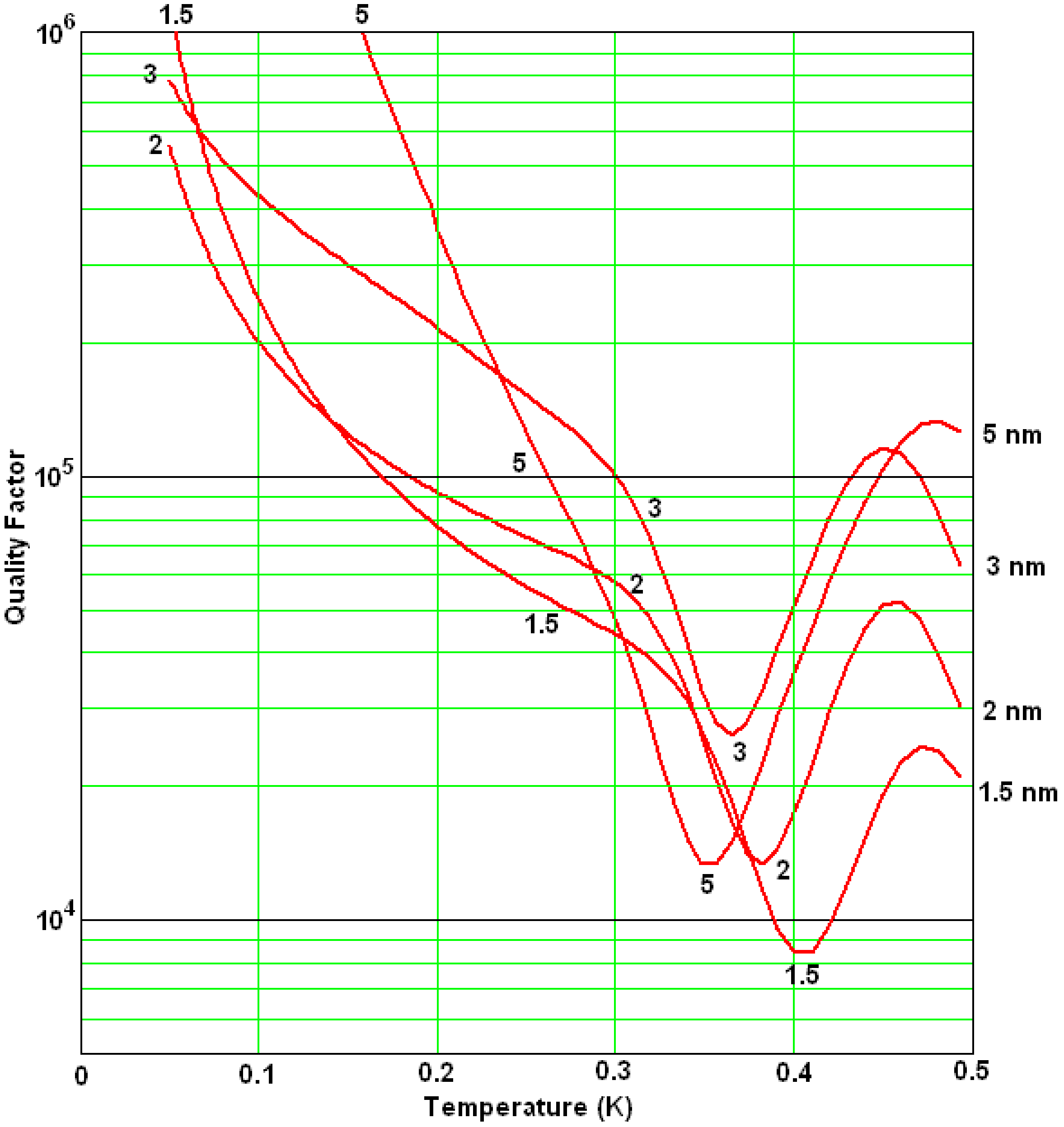

Thermal-toThickness Oscillation Ratio


Third Sound Speed

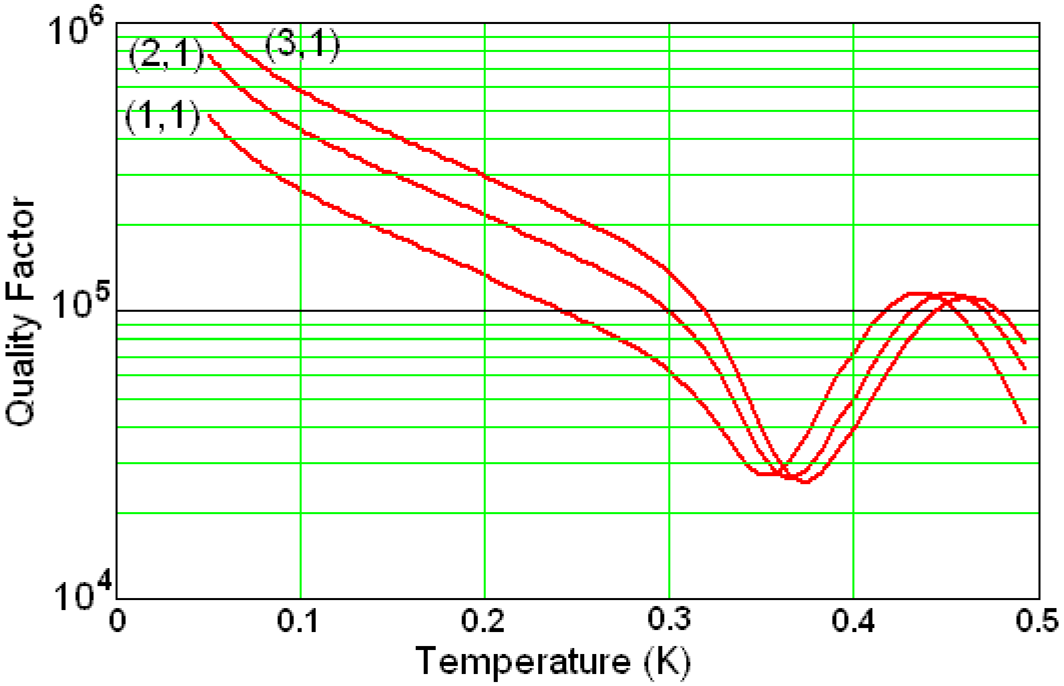

Quality Factor

Model for fixed $\gamma_s = 1000$ $\gamma_{netx} = 0$

Q vs. T (Thickness)



Q vs. T (Modes)



Quality Factor
Temperature (K)
10^6
10^5
10^4
0
0.1
0.2
0.3
0.4
0.5
(1,1)
(2,1)
(3,1)